$y_1 = e^{-x}$ is a solution of $xy'' + (1+2x)y' + (1+x)y = 0$.
Find a second linearly independent solution.

ALL ITEMS IN ALL QUESTIONS WORTH 1 POINT UNLESS OTHERWISE NOTED

SCORE: _____ / 8 PTS

$y_2 = ve^{-x}$

$y_2' = v'e^{-x} - ve^{-x} = (v'-v)e^{-x}$

$y_2'' = (v''-v')e^{-x} - (v'-v)e^{-x}$
$= (v''-2v'+v)e^{-x}$

$xy_2'' + (1+2x)y_2' + (1+x)y_2$
$= [\ xv'' - 2xv' + xv$
$+ (1+2x)v' - (1+2x)v$
$+ (1+x)v]e^{-x}$

$= (xv'' + v')e^{-x} = 0$

$xv'' + v' = 0$

LET $u = v'$ $\quad xu' + u = 0$

$x\frac{du}{dx} + u = 0$

$\frac{1}{u}du = -\frac{1}{x}dx$ $\frac{1}{2}$

$\ln|u| = -\ln|x|$ $\frac{1}{2}$

$v' = u = x^{-1}$ $\frac{1}{2}$

$v = \ln|x|$ $\frac{1}{2}$

$y_2 = e^{-x}\ln|x|$

Consider the non-homogeneous linear differential equation $\cancel{4}y'' + \cancel{20}y' + \cancel{25}y = A\cos 2x$.

SCORE: _____ / 6 PTS

[a] If $y = 2\sin 2x - 3\cos 2x$ is a particular solution of the equation, find the value of A.

$y' = 6\sin 2x + 4\cos 2x$ $\frac{1}{2}$

$y'' = -8\sin 2x + 12\cos 2x$ $\frac{1}{2}$

$y'' + y' + y = 13\cos 2x$

$A = 13$

[b] **Using linearity**, find a particular solution of $\cancel{4}y'' + \cancel{20}y' + \cancel{25}y = 5\cos 2x$.

$\frac{1}{2}$

$y = \frac{5}{13}(2\sin 2x - 3\cos 2x) = \frac{10}{13}\sin 2x - \frac{15}{13}\cos 2x$

[c] Find the general solution of $\cancel{4}y'' + \cancel{20}y' + \cancel{25}y = 5\cos 2x$.

$r^2 + r + 1 = 0$

$\frac{1}{2}$ $r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

$y = \frac{10}{13}\sin 2x - \frac{15}{13}\cos 2x + c_1 e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x + c_2 e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x$

Consider the homogeneous linear differential equation $9x^2y'' + 3xy' + By = 0$. SCORE: _____ / 6 PTS

[a] Find the general solution if $B = 1$.

$9r^2 - 6r + 1 = 0$

$r = \frac{1}{3}, \frac{1}{3}$ $\frac{1}{2}$

$y = C_1 x^{\frac{1}{3}} + C_2 x^{\frac{1}{3}} \ln x$

[b] Find the general solution if $B = 5$.

$9r^2 - 6r + 5 = 0$

$r = \frac{1}{3} \pm \frac{2}{3}i$ $\frac{1}{2}$

$y = C_1 x^{\frac{1}{3}} \cos(\frac{2}{3} \ln x) + C_2 x^{\frac{1}{3}} \sin(\frac{2}{3} \ln x)$

Find the general solution of the homogeneous linear differential equation $2y''' - 11y'' + 6y' + 6y = 0$. SCORE: _____ / 6 PTS

$2r^3 - 11r^2 + 6r + 6 = 0$

0 OR 2 POSITIVE ROOTS

1 NEGATIVE ROOT

$r = \pm \frac{1, 2, 3, 6}{1, 2} = \pm 1, 2, 3, 6, \frac{1}{2}$

$$\begin{array}{r|rrrr} -1 & 2 & -11 & 6 & 6 \\ & & -2 & 13 & -19 \\ \hline & 2 & -13 & 19 & -13 \end{array}$$

← ALTERNATING – TOO NEGATIVE

$$\begin{array}{r|rrr|r} -\frac{1}{2} & 2 & -11 & 6 & 6 \\ & & -1 & 6 & -6 \\ \hline & 2 & -12 & 12 & 0 \end{array}$$

$(r + \frac{1}{2})(2r^2 - 12r + 12) = 0$

$(2r + 1)(r^2 - 6r + 6) = 0$

$r = 3 \pm \sqrt{3}$

$y = C_1 e^{-\frac{1}{2}x} + \boxed{C_2 e^{(3+\sqrt{3})x} + C_3 e^{(3-\sqrt{3})x}}$

A homogeneous linear differential equation with constant coefficients has characteristic polynomial $r(r^2 - 4r + 7)^2$. Find the general solution. SCORE: _____ / 4 PTS

$r = 0, 2 \pm \sqrt{3}i, 2 \pm \sqrt{3}i$

$y = C_1 + C_2 e^{2x} \cos \sqrt{3}x + C_3 e^{2x} \sin \sqrt{3}x$

$+ C_4 x e^{2x} \cos \sqrt{3}x + C_5 x e^{2x} \sin \sqrt{3}x$